

# Convergence of Binomial Tree Method and Explicit Difference Schemes for American Put Options with Time Dependent Coefficients

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**Abstract** Convergence of binomial tree method and explicit difference schemes for the variational inequality model of American put options with time dependent coefficients is studied. When volatility is time dependent, it is not reasonable to assume that the dynamics of the underlying asset's price forms a binomial tree if a partition of time interval with equal parts is used. A time interval partition method that allows binomial tree dynamics of the underlying assets price is provided. Conditions under which the prices of American put option by BTM and explicit difference scheme have the monotonicity on time variable are found. Convergence of explicit difference schemes for variational inequality model of American put options to viscosity solution is proved.

**Keywords** American put option; binomial tree method; monotonicity; convergence; viscosity solution; time dependent coefficient.

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## 1 Introduction

There are two kinds of numerical methods for option pricing; one is based on the probabilistic approach and another one is the finite difference method for PDE.

The binomial tree method (BTM), first proposed by Cox, Ross and Rubinstein [6], is one of the probabilistic numerical methods for pricing options. Due to its simplicity and flexibility, it has become one of the most popular approaches to pricing options. [1, 2, 5, 10, 14, 18, 19]

It is well known that the BTM for European option in Black - Scholes diffusion model converges to the corresponding continuous time model of Black and Scholes ([8]). In particular, Jiang [10] showed that the BTM for European option is equivalent to a special explicit finite difference scheme for Black-Scholes PDE and proved its convergence using PDE approach.

Amin and Khanna [2] first proved the convergence of BTM for American options using probabilistic approach.

Jiang and Dai ([11, 12]) proved the convergence of explicit difference scheme and BTM for American options using viscosity solution theory of PDE. They showed that the BTM for American option is equivalent to a special explicit finite difference scheme for a variational inequality related to Black-Scholes PDE, proved monotonic property of the price by BTM and explicit finite difference scheme, existence and monotones of approximated optimal exercise boundary and used the method of Barles et al [3, 4] and comparison principle of [3, 7]. Jiang and Dai [13] studied the convergence of BTM for European and American path dependent options by PDE approach. Liang et al [16] obtained a convergence rate of the BTM for American put options with penalty method and Hu et al [9] obtained an

optimal convergence rate for an explicit finite difference scheme and BTM for a variational inequality problem of American put options.

BTM is extended to the jump-diffusion models for option pricing. Amin [1] generalized their algorithm of [2] to jump-diffusion models. Zhang [25] studied numerical analysis for American option in jump-diffusion models. Xu et al [24] studied numerical analysis for BTM for European options in Amin's jump-diffusion models and gave strict error estimation for explicit difference scheme and optimal error estimation for BTM. Qian et al [23] proved equivalence of BTM and explicit difference scheme for American option in jump-diffusion models, convergence of explicit difference scheme, existence and monotones of optimal exercise boundary. Luo [19] studied approximated optimal exercise boundary of American option in jump-diffusion model. Liang [15] obtained a convergence rate of the BTM for American put options in jump-diffusion models. Liang et al [17] obtained an optimal convergence rate for BTM for a variational inequality problem of American put options in jump-diffusion models and a convergence rate estimate of approximated optimal exercise boundary to the actual free boundary.

The above all results are obtained under the assumption that the interest rate and volatility are all constants.

On the other hand, Jiang [10] studied Black-Scholes PDE with time dependent coefficients as a model for European options in diffusion model and provided the generalized Black-Scholes formula. H.C. O et al [22] derived a pricing formula of higher order binary with time dependent coefficients and using it, studied the pricing problem of corporate zero coupon bonds. Such higher order binaries with time dependent coefficients are arising in the pricing problem of corporate bonds with discrete coupon ( [21]). H.C. O et al [20] studied some general properties of solutions to inhomogeneous Black-Scholes PDEs with discontinuous maturity payoffs and time dependent coefficients.

This article concerns with binomial tree methods and monotonic properties for American put options with time dependent coefficients. We consider monotonic properties and convergences of prices by binomial tree methods and explicit difference schemes for the variational inequality model of American put options with time dependent coefficients and then using them prove the decreasing property of the price of American put options and increasing property of the optimal exercise boundary on time variable.

When the coefficients are time dependent, in particular, in the case with time dependent volatility, it is not reasonable to assume that the dynamics of the underlying assets price forms a binomial tree if we use a partition of time interval with equal parts. Thus one of our main problems is to find a time interval partition method that allows binomial tree dynamics of the underlying assets price. Another point is to prove the monotonic property of option price and approximated optimal exercise boundary. Jiang and Dais convergence proof ( [12]) strongly depends on the monotonic property of option price but such monotonic property of option price may not hold when coefficients including interest rate and volatility are time dependent as you can see in the following remark 4. We found a special time interval partition method and conditions under which the prices of American put option by BTM and explicit difference scheme have the monotonic property on time variable. Such a special partition of time interval needs some annoying consideration in proving convergence to viscosity solutions.

The remainder of this article is organized as follows. In section 2, we find a time interval partition method that allows binomial tree dynamics of the underlying assets price and

briefly mention BTM for European options. In section 3, we study BTM price of American put option, its monotonic property and existence of approximate optimal exercise boundary. In section 4, we study explicit difference scheme for variational inequality model for American put option and show the monotonicity of option price on time-variable and existence of approximated optimal exercise boundary. Section 5 is devoted to the convergence proof of the explicit difference scheme and BTM.

## 2 Time Interval Partition and BTM for European Options with Time dependent coefficients.

Let  $r(t)$ ,  $q(t)$  and  $\sigma(t)$  be the interest rate, the dividend rate and the volatility of the underlying asset of option, respectively. Let  $0 = t_0 < t_1 < \dots < t_N = T$  be a partition of life time interval  $[0, T]$  and denote as follows

$$\begin{aligned} r_n &= r(t_n), q_n = q(t_n), \quad \sigma_n = \sigma(t_n), \\ \eta_n &= 1 + q_n \Delta t_n, \quad \rho_n = 1 + r_n \Delta t_n, \\ \Delta t_n &= t_{n+1} - t_n, \quad n = 1, \dots, N-1. \end{aligned}$$

The volatility  $\sigma(t)$  of the underlying asset determines the fluctuation of its price in time interval  $[t, t + \Delta t]$ . So if we divide  $[0, T]$  by equal parts, then the dynamics of the underlying assets price in subinterval  $[t_n, t_{n+1}]$  of time may not form a binomial tree. It makes BTM difficult in the case with time dependent coefficients.

On the other hand, from the practical meaning of the volatility  $\sigma_n$ , although we consider the underlying assets price  $S$  in the some interval  $[t_n, t_{n+1}]$ , the underlying assets price  $S$  largely changes if  $\sigma_n$  is large; the underlying assets price  $S$  changes a little if  $\sigma_n$  is small. So we can imagine that we can make the widths of changes of  $S$  in all subintervals a constant if we differently define the length  $\Delta t_n = t_{n+1} - t_n$  of subinterval  $[t_n, t_{n+1}]$  according to the size of  $\sigma_n$ . In other words, if we define  $\Delta t_n = t_{n+1} - t_n$  such that  $\sigma_n \cdot \Delta t_n = \text{const} = (\ln u)^2$ , then we can assume that the width of change of  $S$  in every subinterval  $[t_n, t_{n+1}]$  is  $u$  and the dynamics of  $S$  in every subinterval  $[t_n, t_{n+1}]$  satisfies one period - two states model [10]. Then  $S_t$  are random variables and the evolution in  $[0, T]$  forms a binomial tree. Such a partition method provides a key to overcome the difficulty arising in the case with time dependent coefficients.

Let us define  $t_n (n = 1, \dots, N)$  more definitely. Let assume  $u > 1$ . First, we define

$$t_0 = 0, \quad \sigma_0 = \sigma(t_0), \quad \Delta t_0 = \frac{(\ln u)^2}{\sigma_0^2}, \quad t_1 = t_0 + \Delta t_0 = (\ln u)^2 \cdot \frac{1}{\sigma_0^2}.$$

If  $t_1 \leq T$ , then we define as follows:

$$\sigma_1 = \sigma(t_1), \quad \Delta t_1 = \frac{(\ln u)^2}{\sigma_1^2}, \quad t_2 = t_1 + \Delta t_1 = (\ln u)^2 \cdot \left( \frac{1}{\sigma_0^2} + \frac{1}{\sigma_1^2} \right).$$

Inductively, if  $t_n \leq T$ , then we define as follows:

$$\sigma_n = \sigma(t_n), \quad \Delta t_n = \frac{(\ln u)^2}{\sigma_n^2}, \quad t_{n+1} = t_n + \Delta t_n = (\ln u)^2 \cdot \left( \frac{1}{\sigma_0^2} + \dots + \frac{1}{\sigma_n^2} \right). \quad (1)$$

Such a process is continued until  $t_N \leq T < t_{N+1}$ . Then the number  $N$  of subintervals depends on  $u, T$  and  $\sigma(t)$ .

If we assume that

$$0 < \underline{\sigma} \leq \sigma(t) \leq \bar{\sigma}, \quad (2)$$

then we obtain lower and upper bounds for the size  $\Delta t_n$  of subintervals of time and the number  $N$  of subintervals. From the definition (1)(2.1) of  $\Delta t_n$ , we have

$$\frac{(\ln u)^2}{\bar{\sigma}^2} \leq \Delta t_n \leq \frac{(\ln u)^2}{\underline{\sigma}^2}. \quad (3)$$

On the other hand, if we use  $t_N \leq T < t_{N+1}$ , then we have

$$\frac{T \cdot \underline{\sigma}^2}{(\ln u)^2} - 1 < N \leq \frac{T \cdot \bar{\sigma}^2}{(\ln u)^2}. \quad (4)$$

**Remark 1.** If  $u \downarrow 1$ , then  $N \rightarrow \infty$  and  $0 \leq T - t_N < \Delta t_N = (\ln u)^2 \cdot \frac{1}{\sigma_N^2} \rightarrow 0$ .

Now we consider the dynamics of the underlying asset's price  $S$ . Assume that the width of change of  $S$  in every subinterval  $[t_n, t_{n+1}]$  is  $u$ ,  $d = u^{-1}$  and the dynamics of  $S$  in every subinterval  $[t_n, t_{n+1}]$  satisfies one period - two states model. That is, the underlying assets price  $S_{t_n}$  at time  $t_n$  is changed into  $S_{t_n} u$  or  $S_{t_n} d$ . If the initial price of  $S$  is  $S_0$ , then  $S_{t_n}$  can take one of the following values

$$S_\alpha^n = S_0 u^{n-\alpha} d^\alpha \quad (0 \leq \alpha \leq n) \quad \text{or} \quad S_j = S_0 u^j \quad (j = n, n-2, \dots, -n+2, -n).$$

Assume that

$$d\eta_n < \rho_n < u\eta_n, \quad n = 0, 1, \dots, N. \quad (5)$$

If we denote

$$\theta_n = \frac{\rho_n/\eta_n - d}{u - d}, \quad n = 0, 1, \dots, N. \quad (6)$$

then we have  $0 < \theta_n < 1$  and BTM price of European option with time dependent coefficients is provided as follows:

$$\begin{aligned} V_\alpha^N &= (S_\alpha^N - E)^+ \quad (\text{for call}) \quad \text{or} \quad (E - S_\alpha^N)^+ \quad (\text{for put}), \quad 0 \leq \alpha \leq N, \\ V_\alpha^n &= \frac{1}{\eta_n} [\theta_n V_{\alpha+1}^{n+1} + (1 - \theta_n) V_{\alpha-1}^{n+1}], \quad 0 \leq \alpha \leq n, \quad n = N-1, \dots, 1, 0. \end{aligned} \quad (7)$$

**Remark 2.** Using Jiang's method ([10]), we can easily prove the followings: BTM can be seen as a special explicit difference scheme for Black-Scholes PDE

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{\sigma^2(t)}{2} \frac{\partial^2 V}{\partial x^2} + \left[ r(t) - q(t) - \frac{\sigma^2(t)}{2} \right] \frac{\partial V}{\partial x} - r(t)V &= 0, \quad -\infty < x < \infty, \quad 0 \leq t < T, \\ V(x, T) &= (e^x - E)^+ \quad \text{or} \quad (E - e^x)^+, \quad -\infty < x < \infty. \end{aligned} \quad (8)$$

Let  $x_m = m\Delta x$  ( $-\infty < m < \infty$ ),  $0 = t_0 < t_1 < \dots < t_N = T$  and  $\Delta t_n = t_{n+1} - t_n$ . Denote  $V_m^n = V(x_m, t_n)$ . Then the explicit difference scheme for (8) is provided as follows:

$$\begin{aligned} V_m^N &= (e^{m\Delta x} - E)^+ \text{ or } (E - e^{m\Delta x})^+, \\ V_m^n &= \frac{1}{1 + r_n \Delta t_n} \left\{ \left( 1 - \frac{\sigma_n^2 \Delta t_n}{\Delta x^2} \right) V_m^{n+1} + \left[ \frac{\sigma_n^2 \Delta t_n}{2\Delta x^2} + \frac{1}{2} \left( r_n - q_n - \frac{\sigma_n^2}{2} \right) \frac{\Delta t_n}{\Delta x} \right] V_{m+1}^{n+1} \right. \\ &\quad \left. + \left[ \frac{\sigma_n^2 \Delta t_n}{2\Delta x^2} - \frac{1}{2} \left( r_n - q_n - \frac{\sigma_n^2}{2} \right) \frac{\Delta t_n}{\Delta x} \right] V_{m-1}^{n+1} \right\}, \quad n = N-1, \dots, 1, 0. \end{aligned} \quad (9)$$

The scheme (9) is consistent if  $r(t)$ ,  $q(t)$  and  $\sigma(t)$  are bounded and continuous on  $[0, T]$ . Such an explicit difference scheme is stable if

$$\sigma_n^2 \Delta t_n \leq \Delta x^2; \quad 1 - \frac{1}{\sigma_n^2} \left| r_n - q_n - \frac{\sigma_n^2}{2} \right| \Delta x \geq 0, \quad 0 \leq \forall n \leq N-1.$$

Let  $\Delta x \rightarrow 0$ . Then  $\Delta t_n \rightarrow 0$  and (9) converges to the solution to (8). So BTM price (7) also converges to the solution to (8).

### 3 BTM for American Put Options with Time Dependent Coefficients.

Let  $0 = t_0 < t_1 < \dots < t_N \leq T$  be the partition of time defined in (1) and let

$$S_j = S_0 u^j \quad (j = n, n-2, \dots, -n+2, -n; \quad n = 0, \dots, N),$$

$$\varphi_j = (E - S_j)^+.$$

Then BTM prices  $V_j^n = V(S_j, t_n)$  of American put option are provided as follows:

$$\begin{aligned} V_j^N &= \varphi_j, \\ V_j^n &= \max \left\{ \frac{1}{\rho_n} [\theta_n V_{j+1}^{n+1} + (1 - \theta_n) V_{j-1}^{n+1}], \varphi_j \right\}, \quad n = N-1, \dots, 1, 0. \end{aligned} \quad (10)$$

Now we consider the monotonic property of BTM price  $V_j^n$  for American put option.

**Theorem 1** *BTM prices of American put option*

$$V_j^n = P(S_j, t_n; E) \quad (n = 0, 1, \dots, N, j = n, n-2, \dots, -n+2, -n) \quad (11)$$

are decreasing with respect to  $S_j$  and increasing with respect to  $E$ . That is,

$$\begin{aligned} V_j^n &= P(S_j, t_n; E) \geq P(S_{j+1}, t_n; E) = V_{j+1}^n, \\ P(S_j, t_n; E_1) &\leq P(S_j, t_n; E_2) \quad \text{if } E_1 < E_2. \end{aligned}$$

**Proof**  $V_j^N = \varphi_j = (E - S_j)^+$  is decreasing function on  $S_j$  and increasing on  $E$ . Now assume that  $V_j^{k+1} \geq V_{j+1}^{k+1}$  when  $n = k + 1$ . Then we have

$$\begin{aligned} V_j^k &= \max \left\{ \frac{1}{\rho_k} [\theta_k V_{j+1}^{k+1} + (1 - \theta_k) V_{j-1}^{k+1}], \varphi_j \right\} \\ &\geq \max \left\{ \frac{1}{\rho_k} [\theta_k V_{j+2}^{k+1} + (1 - \theta_k) V_j^{k+1}], \varphi_{j+1} \right\} = V_{j+1}^k. \end{aligned}$$

Thus  $V_j^k$  is decreasing on  $S_j$ . Similarly, we can prove  $V_j^k$  is increasing on  $E$ . (QED)

In order to prove that  $V_j^n$  is decreasing on time variable, we need the following lemmas.

**Lemma 1** (i) If  $r(t)/\sigma^2(t)$  is increasing on  $t$ , then  $\rho_n \leq \rho_{n+1}$ .  
(ii) If  $q(t)/\sigma^2(t)$  is decreasing on  $t$ , then  $\eta_n \geq \eta_{n+1}$ .  
(iii) If  $r(t)/\sigma^2(t)$  is increasing,  $q(t)/\sigma^2(t)$  decreasing and  $\Delta t_n$  is sufficiently small, then

$$\rho_n/\eta_n \leq \rho_{n+1}/\eta_{n+1} ; \theta_n \leq \theta_{n+1}.$$

**Proof** (i) If  $r(t)/\sigma^2(t)$  is increasing, then from the definition of  $\Delta t_n$ , we have

$$\begin{aligned} \frac{r_{n+1}}{\sigma_{n+1}^2} \geq \frac{r_n}{\sigma_n^2} &\Leftrightarrow (\ln u)^2 \cdot \frac{r_{n+1}}{\sigma_{n+1}^2} \geq (\ln u)^2 \cdot \frac{r_n}{\sigma_n^2} \Leftrightarrow \\ \rho_{n+1} = 1 + r_{n+1} \Delta t_{n+1} &\geq 1 + r_n \Delta t_n = \rho_n. \end{aligned}$$

(ii) is proved in similar way with (i).

(iii) If  $\Delta t_n$  is sufficiently small, then  $\eta_n > 0$ . Since  $\rho_n \leq \rho_{n+1}$  and  $\eta_n \geq \eta_{n+1}$ , we have  $\rho_n/\eta_n \leq \rho_{n+1}/\eta_{n+1}$ . Thus from (6), we have  $\theta_n \leq \theta_{n+1}$ . (QED)

**Lemma 2** (i) If  $A \leq B$  and  $0 \leq \alpha \leq \beta$ , then  $\alpha A + (1 - \alpha)B \geq \beta A + (1 - \beta)B$ .

**Proof**  $\alpha A + (1 - \alpha)B - \beta A - (1 - \beta)B = (\beta - \alpha)(B - A) \geq 0$ . (QED)

**Theorem 2** Assume that (5) is satisfies,  $r(t)/\sigma^2(t)$  is increasing and  $q(t)/\sigma^2(t)$  decreasing on  $t$ . Then for BTM prices  $V_j^n$  of American put option we have

$$V_j^{n-1} \geq V_j^n.$$

**Proof** From (10) we have

$$V_j^{N-1} \geq \varphi_j = V_j^N \quad (j = N, N-2, \dots, -N+2, -N).$$

Now assume that  $V_j^k \geq V_j^{k+1}$  ( $\forall j$ ). Then we have

$$\begin{aligned}
 V_j^{k-1} &= \max \left\{ \frac{1}{\rho_{k-1}} [\theta_{k-1} V_{j+1}^k + (1 - \theta_{k-1}) V_{j-1}^k], \varphi_j \right\} \\
 &\geq \max \left\{ \frac{1}{\rho_{k-1}} [\theta_{k-1} V_{j+1}^{k+1} + (1 - \theta_{k-1}) V_{j-1}^{k+1}], \varphi_j \right\} \\
 &\geq \max \left\{ \frac{1}{\rho_k} [\theta_{k-1} V_{j+1}^{k+1} + (1 - \theta_{k-1}) V_{j-1}^{k+1}], \varphi_j \right\} \\
 &\geq \max \left\{ \frac{1}{\rho_k} [\theta_k V_{j+1}^{k+1} + (1 - \theta_k) V_{j-1}^{k+1}], \varphi_{j+1} \right\} = V_j^k.
 \end{aligned}$$

Here the first inequality comes from the induction assumption  $V_j^k \geq V_j^{k+1}$  ( $\forall j$ ), the second inequality from lemma 1 (i), the last inequality from lemma 1 (iii), theorem 1 and lemma 2. (QED)

**Remark 3.** Theorem 2 strongly represents the effect of time dependent coefficients. Here the main tools are lemma 1 and lemma 2.

**Remark 4.** The conditions of theorem 2 are essential. See the following figures:

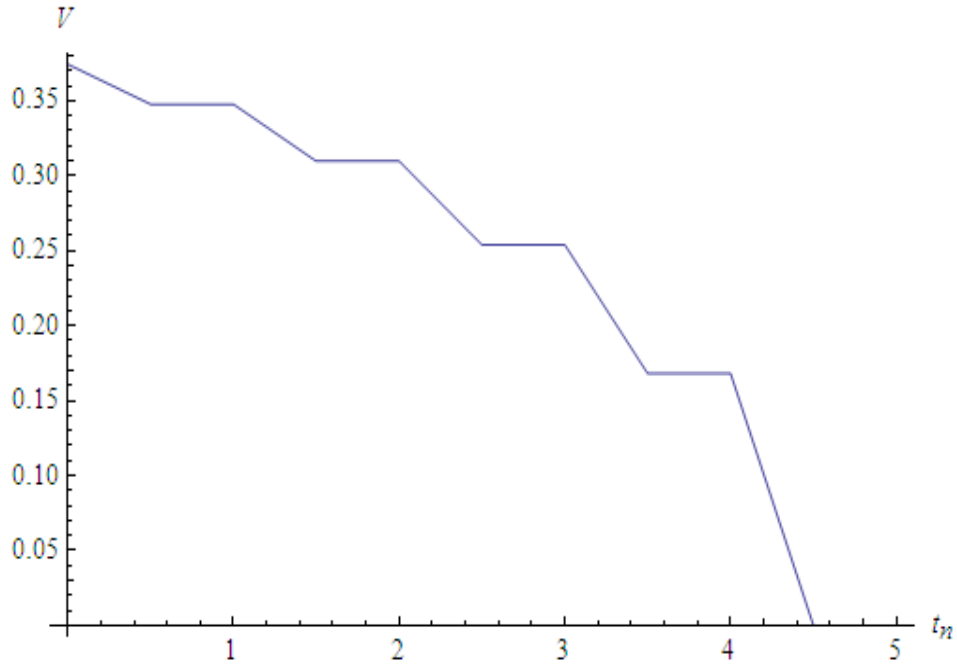


Figure 1: Plot  $(t_n : V(S_j, t_n))$  when  $r(t) = 0.1, q = 0, \sigma = 1, T = 5, E = 1, j = 1$

**Remark 5.** Only using the analogs of lemma 1 and lemma 2, it seems difficult to prove

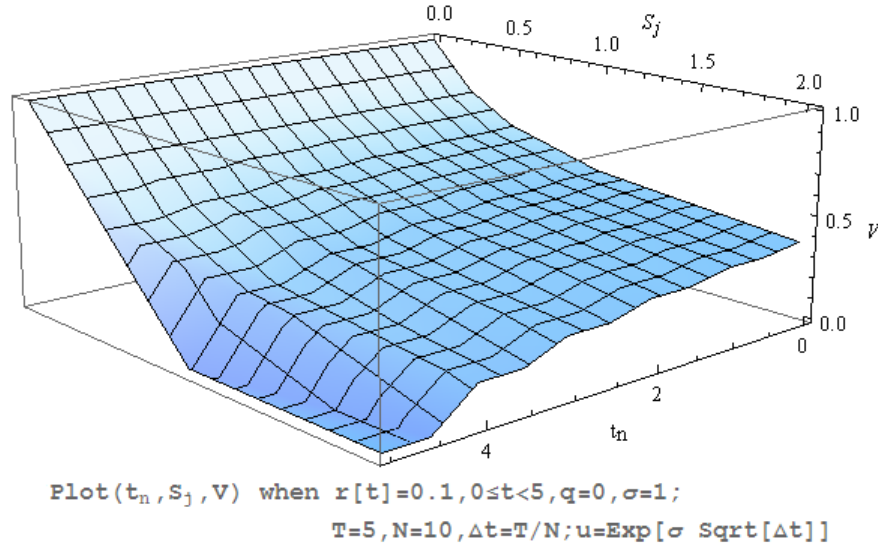


Figure 2:  $r(t)$  is increasing, so  $V$  is decreasing on  $t$ .

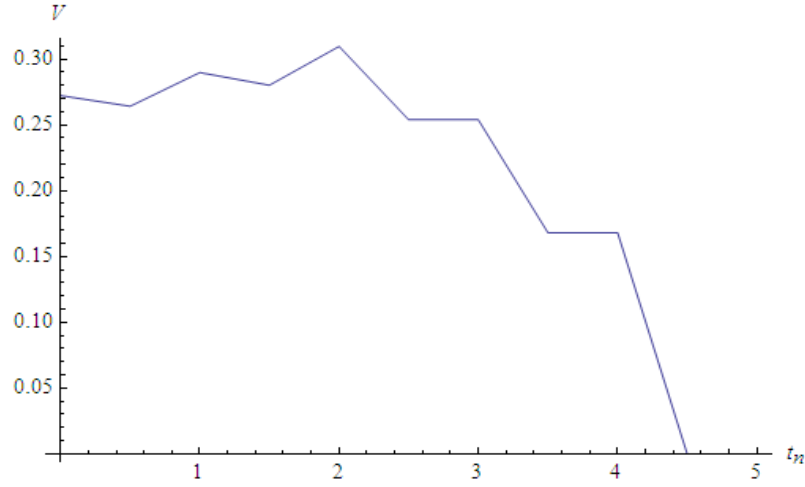
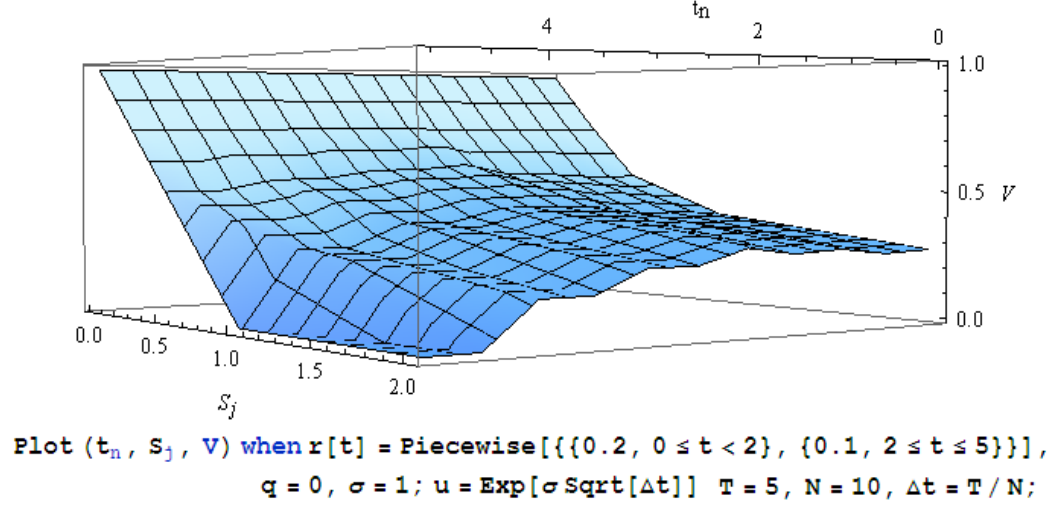


Figure 3: Plot ( $t_n : V(S_j, t_n)$ ) when  $r(t) = \text{Piecewise}\{\{0.2, 0 \leq t < 2\}, \{0.1, 2 \leq t < 5\}\}; q=0, \sigma=1, T=5, E=1, j=1$



Figure 4:  $r(t)$  is not increasing, so  $V$  is not decreasing on  $t$ 

that American call option's BTM price is decreasing on  $t$ .

Now we consider the existence of approximated optimal exercise boundary.

**Theorem 3** *Let  $\Delta t_n$  be sufficiently small. Under the conditions of theorem 2, for every  $t_n$  ( $0 \leq n \leq N-1$ ), there exists a  $j_n \in Z$  such that*

$$\begin{aligned} V_j^n &= \varphi_j \text{ for } j \leq j_n, \\ V_j^n &> \varphi_j \text{ for } j = j_n + 1, \\ V_j^n &\leq \varphi_j \text{ for } j \geq j_n + 2. \end{aligned} \quad (12)$$

Furthermore we have

$$j_{n-1} \leq j_n. \quad (13)$$

**Proof** Without loss of generality, we assume that  $S_0 = 1$  and  $E = 1$ . (Otherwise, use change of variables  $\hat{S} = S/E$ ,  $\hat{h} = V/E$ .) Since

$$V_j^N = (1 - S_j)^+ = \varphi_j, \quad (j = N, N-2, \dots, -N+2, -N),$$

we have

$$\varphi_j = V_j^N = 0 \quad (j \geq 0); \quad \varphi_j = V_j^N > 0, \quad (j \leq -1).$$

Since  $V_j^{N-1} = \max \left\{ \frac{1}{\rho_{N-1}} [\theta_{N-1} \varphi_{j+1} + (1 - \theta_{N-1}) \varphi_{j-1}], \varphi_j \right\}$ , we have

$$V_j^{N-1} \geq 0 = V_j^N = \varphi_j, \quad j \geq 0.$$

In particular,

$$V_j^{N-1} = \varphi_j \ (j \geq 1); \ V_0^{N-1} = \rho_{N-1}^{-1}(1 - \theta_{N-1})\varphi_{-1} > 0 = \varphi_0. \quad (14)$$

Now we consider the case of  $j \leq -1$ .

$$\begin{aligned} V_j^{N-1} &= \max \left\{ \frac{1}{\rho_{N-1}} [\theta_{N-1}\varphi_{j+1} + (1 - \theta_{N-1})\varphi_{j-1}], \varphi_j \right\} \\ &= \max \left\{ \frac{1}{\rho_{N-1}} [\theta_{N-1}(1 - u^{j+1}) + (1 - \theta_{N-1})(1 - u^{j-1})], \varphi_j \right\} \\ &= \max \{ \rho_{N-1}^{-1} - \eta_{N-1}^{-1}u^j, 1 - u^j \}. \end{aligned} \quad (15)$$

Note that if  $j \rightarrow -\infty$ , then  $\rho_{N-1}^{-1} - \eta_{N-1}^{-1}u^j < 1$  and  $1 - u^j \rightarrow 1$ . So there exists

$$j_{N-1} = \max\{j \in Z : j \leq -1, \rho_{N-1}^{-1} - \eta_{N-1}^{-1}u^j \leq 1 - u^j\}.$$

If  $j \leq j_{N-1}$ , then we have  $\rho_{N-1}^{-1} - \eta_{N-1}^{-1}u^j \leq 1 - u^j$  and thus  $V_j^{N-1} = 1 - u^j = \varphi_j$ . If  $j_{N-1} + 1 \leq j \leq -1$ , then we have  $\rho_{N-1}^{-1} - \eta_{N-1}^{-1}u^j > 1 - u^j$  and thus  $V_j^{N-1} > \varphi_j$ . So  $j_{N-1}$  satisfies (12) with  $n = N - 1$ . (In particular if  $\eta_{N-1} \leq 1 (\Leftrightarrow q_{N-1} \leq 0)$ , then  $j_{N-1} = -1$ .)

Now assume that when  $n = k$ , there exists  $j_k$  satisfying (12) and (13). Then if  $j \leq j_k - 1$ , then  $V_{j-1}^k = \varphi_{j-1}$ ,  $V_{j+1}^k = \varphi_{j+1}$  and thus from the formula (10) and the same calculation in (15) we have

$$V_j^{k-1} = \max \left\{ \frac{1}{\rho_{k-1}} [\theta_{k-1}\varphi_{j+1} + (1 - \theta_{k-1})\varphi_{j-1}], \varphi_j \right\} = \max \left\{ \frac{1}{\rho_{k-1}} - \frac{u^j}{\eta_{k-1}}, 1 - u^j \right\}.$$

In the case that  $\eta_{k-1} > 1 (\Leftrightarrow q_{k-1} > 0)$ , let

$$l = \max\{j \in Z : j \leq j_k - 1, \rho_{k-1}^{-1} - \eta_{k-1}^{-1}u^j \leq 1 - u^j\},$$

then we have  $l \leq j_k - 1$ . If  $l < j_k - 1$ , then we define  $j_{k-1} = l$ . Then using the similar way with the consideration when  $n = N - 1$  and theorem 2, we have

$$j \leq j_{k-1} \Rightarrow V_j^{k-1} = \varphi_j; \ j = j_{k-1} + 1 \Rightarrow V_j^{k-1} > \varphi_j; \ j \geq j_{k-1} + 2 \Rightarrow V_j^{k-1} \geq V_j^k \geq \varphi_j.$$

If  $l = j_k - 1$  (that is,  $V_j^{k-1} = \varphi_j$  for all  $j \leq j_k - 1$ ), then note that

$$j = j_{k-1} + 1 \Rightarrow V_j^{k-1} \geq V_j^k > \varphi_j; \ j \geq j_{k-1} + 2 \Rightarrow V_j^{k-1} \geq V_j^k \geq \varphi_j.$$

Generally, we have  $V_j^k \geq \varphi_j$  when  $j = j_k$ . So if  $V_{j_k}^k > \varphi_{j_k}$ , then we define  $j_{k-1} = j_k - 1$ . If  $V_{j_k}^k = \varphi_{j_k}$ , then we define  $j_{k-1} = j_k$ . Thus in any case  $j_{k-1} (\leq j_k)$  is well defined. (QED)

#### 4 Explicit Difference Scheme for Variational Inequality Model of American Options with Time Dependent Coefficients.

A Variational Inequality pricing model of American option with time dependent coefficients is provided as follows:

$$\begin{aligned} \min \left\{ -\frac{\partial V}{\partial t} - \frac{\sigma(t)^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} - (r(t) - q(t)) S \frac{\partial V}{\partial S} + r(t) V, V - \psi \right\} &= 0, \\ 0 \leq t < T, \quad 0 < S < \infty, \\ V(S, T) &= \psi(S), \quad 0 < S < \infty. \end{aligned} \quad (16)$$

Here

$$\psi(S) = (S - E)^+ \quad (\text{for call}), \quad \psi(S) = (E - S)^+ \quad (\text{for put}).$$

Using the transformation

$$u(x, t) = V(S, t); \quad S = e^x, \quad (17)$$

the problem (16) is changed to the following problem

$$\begin{aligned} \min \left\{ -\frac{\partial u}{\partial t} - \frac{\sigma(t)^2}{2} \frac{\partial^2 u}{\partial x^2} - \left( r(t) - q(t) - \frac{\sigma(t)^2}{2} \right) \frac{\partial u}{\partial x} + r(t) u, u - \varphi \right\} &= 0, \\ 0 \leq t < T, \quad -\infty < x < \infty, \\ u(x, T) &= \varphi(x), \quad -\infty < x < \infty. \end{aligned} \quad (18)$$

Here

$$\varphi(x) = (e^x - E)^+ \quad (\text{for call}), \quad \varphi(x) = (E - e^x)^+ \quad (\text{for put}).$$

We construct a lattice on  $\Sigma = \{-\infty < x < \infty, 0 \leq t < T\}$  as follows: Select any  $c \in R$  and  $\Delta x$ . Let  $x_j = j\Delta x + c$ . When  $0 < \alpha \leq 1$ , we define as follows:

$$\begin{aligned} t_0 &= 0, \quad \Delta t_0 = \frac{\alpha \Delta x^2}{\sigma^2(t_0)}, \quad t_1 = t_0 + \Delta t_0, \quad \Delta t_1 = \frac{\alpha \Delta x^2}{\sigma^2(t_1)}, \dots, \\ t_n &= t_{n-1} + \Delta t_{n-1}, \quad \Delta t_n = \frac{\alpha \Delta x^2}{\sigma^2(t_n)}, \quad n = 0, 1, 2, \dots \end{aligned} \quad (19)$$

This process is continued until  $t_N$  such that

$$t_N = t_{N-1} + \Delta t_{N-1} \leq T < t_{N+1} = t_N + \frac{\alpha \Delta x^2}{\sigma^2(t_N)}.$$

Then we have a lattice on  $\Sigma = \{-\infty < x < \infty, 0 \leq t < T\}$ :

$$Q_c = \{(x_j, t_n) : x_j = j\Delta x + c, 0 \leq n \leq N, j \in Z\}. \quad (20)$$

Under the assumption (2) we have

$$\frac{\alpha \Delta x^2}{\bar{\sigma}^2} \leq \Delta t_n \leq \frac{\alpha \Delta x^2}{\underline{\sigma}^2}. \quad (21)$$

Thus there exists  $N$  such that  $t_N \leq T < t_{N+1}$  and we have

$$\frac{T \cdot \underline{\sigma}^2}{\alpha \Delta x^2} - 1 < N \leq \frac{T \cdot \bar{\sigma}^2}{\alpha \Delta x^2}. \quad (22)$$

Therefore if  $\Delta x \rightarrow 0$ , then  $N \rightarrow \infty$  and  $0 \leq T - t_N \leq \Delta t_N \leq \frac{\alpha \Delta x^2}{\bar{\sigma}^2} \rightarrow 0$ .

$u_j^n = u(j\Delta x + c, t_n)$  represents the value of approximation at  $(j\Delta x + c, t_n)$  and let  $\varphi_j = \varphi(j\Delta x + c)$ . Taking explicit difference for time and the conventional difference discretization for space variable in (18), we have

$$\min \left\{ -\frac{u_j^{n+1} - u_j^n}{\Delta t_n} - \frac{\sigma^2(t_n)}{2} \cdot \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x^2} \right. \\ \left. - \left[ r(t_n) - q(t_n) - \frac{\sigma^2(t_n)}{2} \right] \frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2\Delta x} + r(t_n)u_j^n, u_j^n - \varphi_j \right\} = 0. \quad (23)$$

If we denote  $r_n = r(t_n)$ ,  $q_n = q(t_n)$ ,  $\sigma_n = \sigma(t_n)$ , then (23) is equivalent to

$$u_j^n = \max \left\{ \frac{1}{1 + r_n \Delta t_n} \left\{ \left( 1 - \frac{\sigma_n^2 \Delta t_n}{\Delta x^2} \right) u_j^{n+1} + \frac{\sigma_n^2 \Delta t_n}{\Delta x^2} \left[ \left( \frac{1}{2} + \frac{\Delta x}{2\sigma_n^2} \left( r_n - q_n - \frac{\sigma_n^2}{2} \right) \right) u_{j+1}^{n+1} \right. \right. \right. \right. \\ \left. \left. \left. + \left( \frac{1}{2} - \frac{\Delta x}{2\sigma_n^2} \left( r_n - q_n - \frac{\sigma_n^2}{2} \right) \right) u_{j-1}^{n+1} \right] \right\}, \varphi_j \right\}.$$

Here, if we denote  $S_0 = e^c$ , then

$$\varphi_j = (S_0 e^{j\Delta x} - E)^+ \quad (\text{for call}), \quad \varphi_j = (E - S_0 e^{j\Delta x})^+ \quad (\text{for put}).$$

From (19) we have  $\alpha = \frac{\sigma_n^2 \Delta t_n}{\Delta x^2}$  and let

$$a_n = \frac{1}{2} + \frac{\Delta x}{2\sigma_n^2} \left( r_n - q_n - \frac{\sigma_n^2}{2} \right). \quad (24)$$

Then we have the explicit difference scheme

$$U_j^N = \varphi_j, \quad j \in Z, \quad (25)$$

$$U_j^n = \max \left\{ \frac{1}{\rho_n} \left\{ (1 - \alpha) U_j^{n+1} + \alpha [a_n U_{j+1}^{n+1} + (1 - a_n) U_{j-1}^{n+1}] \right\}, \varphi_j \right\}, \quad (26)$$

$$n = N - 1, \dots, 1, 0.$$

(Note that  $\rho_n = 1 + r_n \Delta t_n$ .) In particular, if  $\alpha = \frac{\sigma_n^2 \Delta t_n}{\Delta x^2} = 1$ , then  $\Delta x = \sigma_n \sqrt{\Delta t_n}$  and

$$U_j^n = \max \left\{ \frac{1}{\rho_n} [a_n U_{j+1}^{n+1} + (1 - a_n) U_{j-1}^{n+1}], \varphi_j \right\}, \quad n = N - 1, \dots, 1, 0. \quad (27)$$

Now we consider the relation of BTM and explicit difference scheme for American option.

**Lemma 3** If  $\ln u = \Delta x = \sigma_n \sqrt{\Delta t_n}$ , we have

$$\theta_n = \frac{1}{2} + \frac{\Delta x}{2\sigma_n^2} \left( r_n - q_n - \frac{\sigma_n^2}{2} \right) + O(\Delta x^3).$$

Here  $\theta_n$  are coefficients of BTM defined by (6).

The proof is easy.

Contrasting (10) and (27), BTM is equivalent to a special explicit difference scheme (27) in the sense of neglecting  $O(\Delta x^3)$ .

Now we show the conditions for American (put) option price be monotonic.

**Theorem 4** Assume that  $0 < \alpha \leq 1$  and  $\left| \frac{\Delta x}{\sigma_n^2} \left( r_n - q_n - \frac{\sigma_n^2}{2} \right) \right| < 1$ .

(i) If  $\varphi_j = (S_0 e^{j\Delta x} - E)^+$  (call), then  $U_j^n \leq U_{j+1}^n$  and  $0 \leq U_j^n \leq e^{j\Delta x + c}$ .

(ii) If  $\varphi_j = (E - S_0 e^{j\Delta x})^+$  (put), then  $U_j^n \geq U_{j+1}^n$  and  $0 \leq U_j^n \leq E$ .

**Proof** From the assumption we have  $0 < a_n \leq 1$ .

(i)  $U_j^N = \varphi_j = (S_0 e^{j\Delta x} - E)^+ \leq (S_0 e^{(j+1)\Delta x} - E)^+ = \varphi_{j+1} = U_{j+1}^N$ . Now assume that  $U_j^{k+1} \leq U_{j+1}^{k+1}$ . Then we have

$$\begin{aligned} U_j^k &= \max \left\{ \frac{1}{\rho_k} \{ (1 - \alpha) U_j^{k+1} + \alpha [a_k U_{j+1}^{k+1} + (1 - a_k) U_{j-1}^{k+1}] \}, \varphi_j \right\} \\ &\leq \max \left\{ \frac{1}{\rho_k} \{ (1 - \alpha) U_{j+1}^{k+1} + \alpha [a_k U_{j+2}^{k+1} + (1 - a_k) U_j^{k+1}] \}, \varphi_{j+1} \right\} = U_{j+1}^k. \end{aligned}$$

(ii) is proved in the same way as (i). (QED)

**Theorem 5** Assume that  $0 < \alpha \leq 1$ ,  $\left| \frac{\Delta x}{\sigma_n^2} \left( r_n - q_n - \frac{\sigma_n^2}{2} \right) \right| < 1$ ,  $r(t)/\sigma^2(t)$  is increasing and  $q(t)/\sigma^2(t)$  decreasing on  $t$ . Then prices  $U_j^n$  of American put option given by (25) and (26) with  $\varphi_j = (E - S_0 e^{j\Delta x})^+$  are decreasing on  $t$ , that is,

$$U_j^n \geq U_j^{n+1}, \quad j \in Z, \quad n = N - 1, \dots, 1, 0.$$

**Proof** When  $n = N - 1$ , from (26) we have  $U_j^{N-1} \geq \varphi_j = U_j^N$ ,  $j \in Z$ . Assume that  $U_j^{k+1} \geq U_j^{k+2}$ ,  $j \in Z$ . From the assumption and lemma 1 (i) we have  $\rho_k \leq \rho_{k+1}$  and thus

$$\begin{aligned} U_j^k &= \max \left\{ \frac{1}{\rho_k} \{ (1 - \alpha) U_j^{k+1} + \alpha [a_k U_{j+1}^{k+1} + (1 - a_k) U_{j-1}^{k+1}] \}, \varphi_j \right\} \\ &\geq \max \left\{ \frac{1}{\rho_{k+1}} \{ (1 - \alpha) U_j^{k+2} + \alpha [a_k U_{j+1}^{k+2} + (1 - a_k) U_{j-1}^{k+2}] \}, \varphi_j \right\}. \end{aligned}$$

From  $a_k = \frac{1}{2} + \frac{\Delta x}{2} \left( \frac{r_k}{\sigma_k^2} - \frac{q_k}{\sigma_k^2} - \frac{1}{2} \right)$  and the assumption, we have  $a_k \leq a_{k+1}$ . By theorem 4 (ii), we have  $U_{j-1}^{k+2} \geq U_{j+1}^{k+2}$  and thus lemma 2 with  $a_k \leq a_{k+1}$  gives us

$$a_{k+1} U_{j+1}^{k+2} + (1 - a_{k+1}) U_{j-1}^{k+2} \leq a_k U_{j+1}^{k+2} + (1 - a_k) U_{j-1}^{k+2}.$$

Therefore we have

$$U_j^k \geq \max \left\{ \frac{1}{\rho_{k+1}} \{ (1 - \alpha)U_j^{k+2} + \alpha [a_{k+1}U_{j+1}^{k+2} + (1 - a_{k+1})U_{j-1}^{k+2}] \}, \varphi_j \right\} = U_j^{k+1}.$$

(QED)

**Remark 6.** Theorem 5 strongly represents the effect of time dependent coefficients. Here the main tools are lemma 1 (i) and lemma 2. The conditions of theorem 5 are essential. If  $r(t)/\sigma^2(t)$  is not increasing, then the price of American put option by explicit difference scheme might not be decreasing on  $t$  as in remark 4.

**Remark 7.** Only using the analogs of lemma 1 and lemma 2, it seems difficult to prove that American call option's price is decreasing on  $t$ .

Now we show the existence of approximated optimal exercise boundary.

**Theorem 6** *Under the assumptions of theorem 5, for any  $0 \leq n \leq N - 1$ , there exists  $j_n \in Z$  such that*

$$j \leq j_n \Rightarrow U_j^n = \varphi_j; \quad j = j_n + 1 \Rightarrow U_j^n > \varphi_j; \quad j \geq j_n + 2 \Rightarrow U_j^n \geq \varphi_j. \quad (28)$$

$$j_0 \leq j_1 \leq \dots \leq j_{N-1}. \quad (29)$$

**Proof** Note that  $U_j^N = (E - S_0 e^{j\Delta x})^+ = \varphi_j$  is decreasing on  $j \in Z$ . Let

$$k_1 = \max\{j \in Z; E - S_0 e^{j\Delta x} > 0\}. \quad (30)$$

Then if  $j \leq k_1 - 1$  then  $j - 1, j, j + 1 \leq k_1$  and

$$\varphi_{j-1} = E - S_0 e^{j\Delta x - \Delta x}, \quad \varphi_j = E - S_0 e^{j\Delta x}, \quad \varphi_{j+1} = E - S_0 e^{j\Delta x + \Delta x} > 0.$$

Let  $u = e^{\Delta x}$ ,  $d = e^{-\Delta x}$  and  $\psi_j = \frac{E}{\rho_{N-1}} - S_0 u^j \cdot \frac{1 - \alpha + \alpha[a_{N-1}u + (1 - a_{N-1})d]}{\rho_{N-1}}$ , then

$$\begin{aligned} U_j^{N-1} &= \max \left\{ \frac{1}{\rho_{N-1}} [(1 - \alpha)\varphi_j + \alpha(a_{N-1}\varphi_{j+1} + (1 - a_{N-1})\varphi_{j-1})], \varphi_j \right\} \\ &= \max\{\psi_j, \varphi_j\}. \end{aligned}$$

Then we have

$$\lim_{j \rightarrow -\infty} \psi_j = \frac{E}{\rho_{N-1}} < E = \lim_{j \rightarrow -\infty} \varphi_j, \quad (j \leq k_1 - 1).$$

First, we consider the case that  $\psi_j \leq \varphi_j$  ( $\forall j \leq k_1 - 1$ ). If  $j \leq k_1 - 1$  then  $U_j^{N-1} = \varphi_j$ , and if  $j = k_1 + 1$  then  $\varphi_j = \varphi_{j+1} = 0$ ,  $\varphi_{j-1} > 0$  and thus we have

$$U_j^{N-1} = \max \left\{ \frac{\alpha(1 - a_{N-1})}{\rho_{N-1}} \varphi_{j-1}, 0 \right\} > 0 = \varphi_j.$$

If  $j \geq k_1 + 2$ , then from (26) we have  $U_j^{N-1} \geq \varphi_j$ . So if  $U_{j_{k_1}}^{N-1} > \varphi_{j_{k_1}}$ , then we define  $j_{N-1} = k_1 - 1$ ; and if  $U_{j_{k_1}}^{N-1} = \varphi_{j_{k_1}}$ , then we define  $j_{N-1} = k_1$ .

Next, we consider the case that  $\exists j (j \leq k_1 - 1) : \psi_j > \varphi_j$ . We define

$$j_{N-1} = \max\{j < k_1 - 1 : \psi_j \leq \varphi_j\}.$$

Then we have

$$\begin{aligned} j \leq j_{N-1} &\Rightarrow \psi_j \leq \varphi_j \Rightarrow U_j^{N-1} = \varphi_j, \\ j = j_{N-1} + 1 &\Rightarrow \psi_j > \varphi_j \Rightarrow U_j^{N-1} = \psi_j > \varphi_j, \\ j \geq j_{N-1} + 2 &\Rightarrow U_j^{N-1} \geq \varphi_j. \end{aligned}$$

Thus we proved the existence of  $j_{N-1} \leq k_1$ .

Now we assume that when  $n = k + 1$  there exists  $j_{k+1}$  such that

$$\begin{aligned} j_{k+1} &\leq j_{k+2} \leq \dots \leq j_{N-1}, \\ j \leq j_{k+1} &\Rightarrow U_j^{k+1} = \varphi_j, \\ j = j_{k+1} + 1 &\Rightarrow U_j^{k+1} > \varphi_j, \\ j \geq j_{k+1} + 2 &\Rightarrow U_j^{k+1} \geq \varphi_j. \end{aligned} \tag{31}$$

If  $j \leq j_{k+1} - 1$ , then  $j + 1, j, j - 1 \leq j_{k+1}$  and thus  $U_i^{k+1} = \varphi_i$  ( $i = j - 1, j, j + 1$ ). As the above, let  $\psi_j = \frac{E}{\rho_k} - S_0 u^j \cdot \frac{1 - \alpha + \alpha[a_k u + (1 - a_k)d]}{\rho_k}$ . Then by (26) we have

$$\begin{aligned} U_j^k &= \max \left\{ \frac{1}{\rho_k} [(1 - \alpha)\varphi_j + \alpha(a_k \varphi_{j+1} + (1 - a_k)\varphi_{j-1})], \varphi_j \right\} \\ &= \max\{\psi_j, \varphi_j\}. \end{aligned}$$

Note that  $\psi_j < \varphi_j$  for sufficiently large  $j \in Z$ . In the case that  $\psi_j \leq \varphi_j$  ( $\forall j \leq j_{k+1} - 1$ ), we have  $U_j^k = \varphi_j$  for all  $j \leq j_{k+1} - 1$ . From theorem 5 and the inductive assumption (31) we have the fact that  $j = j_{k+1} + 1 \Rightarrow U_j^k \geq U_j^{k+1} > \varphi_j$ ;  $j \geq j_{k+1} + 2 \Rightarrow U_j^k \geq U_j^{k+1} \geq \varphi_j$ . Therefore if  $U_{j_{k+1}}^k > \varphi_{j_{k+1}}$ , then let  $j_k = j_{k+1} - 1$ . If  $U_{j_{k+1}}^k = \varphi_{j_{k+1}}$ , then let  $j_k = j_{k+1}$ . In the case that  $\exists j (j \leq j_{k+1} - 1) : \psi_j > \varphi_j$ , we define  $j_k = \max\{j < j_{k+1} - 1 : \psi_j \leq \varphi_j\}$ . Then

$$\begin{aligned} j \leq j_k &\Rightarrow U_j^k = \varphi_j, \\ j = j_k + 1 &\Rightarrow U_j^k = \psi_j > \varphi_j, \\ j \geq j_k + 2 &\Rightarrow U_j^k \geq \varphi_j. \end{aligned}$$

Thus we proved the existence of  $j_k \leq j_{k+1}$ . (QED)

**Remark 8.** If  $\Delta x$  is enough small, then  $j_k \in [j_{k+1} - 1, j_{k+1}]$ .

Now we estimate the optimal exercise boundary near the maturity.

In the first part of the proof of theorem 6, we proved the existence of  $j_{N-1}$ , the approximated optimal exercise boundary near the maturity. If  $k_1$  is the one defined in (30) and  $S_0 = e^c$ , then  $k_1 = \max\{j \in Z; E - e^{j\Delta x + c} > 0\}$  and for  $j \leq k_1 - 1$ , we have  $\varphi_j = E - e^{j\Delta x + c}$  and let

$$\psi_j = \frac{1}{\rho_{N-1}} [(1 - \alpha)\varphi_j + \alpha(a_{N-1}\varphi_{j+1} + (1 - a_{N-1})\varphi_{j-1})].$$

In the case that  $\psi_j \leq \varphi_j$  ( $\forall j \leq k_1 - 1$ ) we know  $j_{N-1} = k_1 - 1$  or  $k_1$ . Then we have  $E - e^{j_{N-1}\Delta x + c} > 0$ ,  $E - e^{(j_{N-1}+2)\Delta x + c} \leq 0$  and thus we have

$$\ln E - 2\Delta x \leq j_{N-1}\Delta x + c \leq \ln E. \quad (32)$$

In the case that  $\exists j$  ( $j \leq k_1 - 1$ ) :  $\psi_j > \varphi_j$ , by theorem 6, we have

$$j_{N-1} = \max\{j \leq k_1 - 1 : \psi_j - \varphi_j \leq 0\}.$$

By using the definition of  $a_n$  and Taylor expansion, we have

$$a_n e^{\Delta x} + (1 - a_n) e^{-\Delta x} = 1 + \frac{r_n - q_n}{\sigma_n^2} \Delta x^2 + O(\Delta x^4).$$

Then for  $j \leq k_1 - 1$  we have

$$\begin{aligned} \psi_j - \varphi_j &= \\ &= \frac{(1 - \alpha)(E - e^{j\Delta x + 1}) + \alpha [a_{N-1}(E - e^{(j+1)\Delta x + 1}) + (1 - a_{N-1})(E - e^{(j-1)\Delta x + 1})]}{\rho_{N-1}} \\ &\quad - (E - e^{j\Delta x + 1}) = \frac{1}{\rho_{N-1}} \cdot \frac{\sigma_{N-1}^2 \Delta t_{N-1}}{\Delta x^2} \left[ (q_{N-1} e^{j\Delta x + c} - r_{N-1} E) \frac{\Delta x^2}{\sigma_{N-1}^2} + O(\Delta x^4) \right]. \end{aligned}$$

Note that  $E > e^{j\Delta x + c}$  for  $j \leq k_1 - 1$ . If  $q_{N-1} \leq r_{N-1}$ , then  $r_{N-1} E > q_{N-1} e^{j\Delta x + c}$  and therefore if  $\Delta x$  is enough small, then we have  $\psi_j < \varphi_j$  ( $\forall j \leq k_1 - 1$ ). Thus in our case, since  $\exists j$  ( $j \leq k_1 - 1$ ) :  $\psi_j > \varphi_j$ , we must have  $q_{N-1} > r_{N-1}$ . Then for sufficiently small  $\Delta x$ , we have  $j_{N-1} = \max\{j \leq k_1 - 1 : q_{N-1} e^{j\Delta x + c} \leq r_{N-1} E\}$  and therefore  $j_{N-1}\Delta x + c \leq \ln \frac{r_{N-1}}{q_{N-1}} E$ . If  $j = j_{N-1} + 1$ , then  $q_{N-1} e^{j\Delta x + c} > r_{N-1} E$  and thus  $(j_{N-1} + 1)\Delta x + c > \ln \frac{r_{N-1}}{q_{N-1}} E$ . So we have

$$\ln \frac{r_{N-1}}{q_{N-1}} E - \Delta x < j_{N-1}\Delta x + c \leq \ln \frac{r_{N-1}}{q_{N-1}} E.$$

Thus combining this inequality with (32), we have the following theorem which provides an estimate of the approximated optimal exercise boundary near the maturity.

**Theorem 7**  $\ln \min \left( E, \frac{r_{N-1}}{q_{N-1}} E \right) - 2\Delta x \leq j_{N-1}\Delta x + c \leq \ln \min \left( E, \frac{r_{N-1}}{q_{N-1}} E \right).$

For fixed  $\Delta x$ , the *approximated optimal exercise boundary*  $x = \rho_{\Delta x}(t)$  is defined as follows:

$$\rho_{\Delta x}(t) = \frac{t - t_n}{t_{n+1} - t_n} (j_{n+1}\Delta x + c) + \frac{t_{n+1} - t}{t_{n+1} - t_n} (j_n\Delta x + c), \quad t \in [t_n, t_{n+1}], n = 0, \dots, N-2.$$

**Corollary** (i)  $\rho_{\Delta x}(t_{N-1}) \in \left[ \ln \min \left( E, \frac{r_{N-1}}{q_{N-1}} E \right) - 2\Delta x, \ln \min \left( E, \frac{r_{N-1}}{q_{N-1}} E \right) \right]$ .  
(ii)  $\rho_{\Delta x}(t)$  is increasing on  $t$ .



## 5 Convergence of the Explicit Difference Scheme and BTM for American Put Option.

In this section we will prove that the explicit difference scheme (25) and (26) for American put option converges to the viscosity solution to the variational inequality (18) and using it prove the monotonic properties of the price of American put option and its optimal exercise boundary.

We denote by  $l^\infty(Z)$  the Banach space of all bounded two sided sequences with sup norm. In  $l^\infty(Z)$ , we define  $(U_j) \leq (V_j) \Leftrightarrow U_j \leq V_j, \forall j \in Z$ .

For fixed every  $n$ , the two sided sequence  $\mathbf{U}^n = (\dots, U_j, \dots)_{j=-\infty}^\infty$  of American put option's prices  $U_j, \forall j \in Z$  given by (25) and (26) is bounded from theorem 4.

If we denote the right side of (26) by  $(\mathbf{F}_n \mathbf{U}^{n+1})_j$ , then  $\mathbf{F}_n$  defines an operator

$$\mathbf{U}^n := \mathbf{F}_n \mathbf{U}^{n+1} = \{(\mathbf{F}_n \mathbf{U}^{n+1})_j\}_{j=-\infty}^\infty \quad (33)$$

sending the sequence  $\mathbf{U}^{n+1}$  of  $t_{n+1}$ -time prices to the sequence  $\mathbf{U}^n$  of  $t_n$ -time prices. The operator  $\mathbf{F}_n$  depends not only on  $n$  and  $\Delta x$  but also on  $t_n$  and  $\Delta t_n$ .

**Lemma 4** *If  $0 < \alpha \leq 1$ ,  $\left| \frac{\Delta x}{\sigma_n^2} \left( r_n - q_n - \frac{\sigma_n^2}{2} \right) \right| < 1$ , then  $\mathbf{F}_n$  is increasing, that is,*

$$\mathbf{U} \leq \mathbf{V}, \mathbf{U}, \mathbf{V} \in l^\infty(Z) \Rightarrow \mathbf{F}_n \mathbf{U} \leq \mathbf{F}_n \mathbf{V}.$$

(Proof) Noting that from the assumption we have  $1 - \alpha \geq 0$  and  $0 < a_n < 1$ , the required result easily comes from (26). (QED)

**Lemma 5** *If  $\mathbf{U} \in l^\infty(Z)$  and  $\mathbf{K} = (\dots, K, K, K, \dots)$  with  $K \geq 0$ , then*

$$\mathbf{F}_n(\mathbf{U} + \mathbf{K}) \leq \mathbf{F}_n \mathbf{U} + \mathbf{K}.$$

(Proof) Since  $\rho_n > 1$ , we have

$$\begin{aligned} & \mathbf{F}_n(\mathbf{U} + \mathbf{K}) = \\ & = \left( \max \left\{ \frac{1}{\rho_n} [(1 - \alpha)(U_j + K) + \alpha(a_n(U_{j+1} + K) + (1 - a_n)(U_{j-1} + K))], \varphi_j \right\} \right)_{j=-\infty}^\infty \\ & \leq \left( \max \left\{ \frac{1}{\rho_n} [(1 - \alpha)U_j + \alpha(a_n U_{j+1} + (1 - a_n)U_{j-1})], \varphi_j \right\} + \max \left\{ \frac{K}{\rho_n}, 0 \right\} \right)_{j=-\infty}^\infty \\ & \leq \mathbf{F}_n \mathbf{U} + \mathbf{K}. \end{aligned}$$

(QED)

Define the extension function  $u_{\Delta x}(x, t)$  as follows:

$$x \in [(j - 1/2)\Delta x + c, (j + 1/2)\Delta x + c), t \in [t_n, t_{n+1}) \Rightarrow u_{\Delta x}(x, t) := U_j^n, \quad (34)$$

$$x \in [(j - 1/2)\Delta x + c, (j + 1/2)\Delta x + c), t \in [t_N, T) \Rightarrow u_{\Delta x}(x, t) := U_j^N. \quad (35)$$

**Remark 9.** Here  $u_{\Delta x}$  is piecewise continuous function. The following discussion is also true if we define  $u_{\Delta x}$  as a continuous function interpolating the data set  $(j\Delta x + c, t_n; U_j^n)$ .

From Theorem 4, we have

$$0 \leq u_{\Delta x}(x, t) \leq E. \quad (36)$$

Therefore, for every fixed  $t$ , if we denote  $\mathbf{u}_{\Delta x}(\bullet, t) := \{u_{\Delta x}(x, t) : x \in \mathbf{R}\}$ , then we have

$$\mathbf{u}_{\Delta x}(\bullet, t) \in l^\infty(Z).$$

When  $t \in [t_n, t_{n+1})$ ,  $n = 0, \dots, N-1$ , if we define

$$\Delta t = \Delta t(t, \Delta x) := \frac{t - t_n}{t_{n+1} - t_n} \Delta t_{n+1} + \frac{t_{n+1} - t}{t_{n+1} - t_n} \Delta t_n, \quad (37)$$

then  $t + \Delta t \in [t_{n+1}, t_{n+2})$  and thus we have

$$\mathbf{u}_{\Delta x}(\bullet, t) = \mathbf{F}_n \mathbf{u}_{\Delta x}(\bullet, t + \Delta t). \quad (38)$$

$\frac{\Delta t}{\Delta t_n} = \frac{t - t_n}{t_{n+1} - t_n} \cdot \frac{\sigma_{n+1}^2}{\sigma_n^2} + \frac{t_{n+1} - t}{t_{n+1} - t_n} \cdot 1$  is a convex linear combination of  $\frac{\sigma_{n+1}^2}{\sigma_n^2}$  and 1. Thus if  $\sigma(t)$  is continuous, then this ratio converges to 1 when  $\Delta x \rightarrow 0$ . Like this, we proved the following lemma:

**Lemma 6** Assume that  $\sigma(t)$  is continuous. Then  $\lim_{\Delta x \rightarrow 0} \frac{\Delta t(t, \Delta x)}{\Delta t_n} = 1$ .

In order to prove the convergence, we recall the notion of viscosity solutions.

Let  $USC([0, T] \times R)$  ( $LSC([0, T] \times R)$ ) be the space of all upper (lower) semi-continuous functions defined on  $[0, T] \times R$ . If  $u \in USC([0, T] \times R)$  ( $LSC([0, T] \times R)$ ) satisfies the following two conditions, then  $u$  is called the *viscosity subsolution (supersolution)* of the variational inequality (18):

- (i)  $u(x, T) \leq (\geq) \varphi(x)$ ,
- (ii) If  $\Phi \in C^{1,2}([0, T] \times R)$  and  $u - \Phi$  attains its local maximum (minimum) at  $(x, t) \in [0, T] \times R$ , we have

$$\min \left\{ -\frac{\partial \Phi}{\partial t} - \frac{\sigma(t)^2}{2} \frac{\partial^2 \Phi}{\partial x^2} - \left( r(t) - q(t) - \frac{\sigma(t)^2}{2} \right) \frac{\partial \Phi}{\partial x} + r(t)u, u - \varphi \right\}_{(x,t)} \leq (\geq) 0$$

$u \in C([0, T] \times R)$  is called the *viscosity solution* of the variational inequality (18) if it is both viscosity subsolution and viscosity supersolution of (18).

**Lemma 7** (Comparison lemma) [7] If  $u$  and  $v$  are viscosity subsolution and supersolution of (18) and  $|u(x, t)|, |v(x, t)| \leq E$ , then  $u \leq v$ .

**Theorem 8** [7] If  $r(t)$ ,  $q(t)$  are continuous on  $[0, T]$ , then the problem (18) has a unique viscosity solution. Furthermore, there exists an optimal exercise boundary  $\rho(t) : [0, T] \rightarrow R$  (continuous function) such that if  $x < \rho(t)$ , then  $u(x, t) = \varphi(x)$ ; if  $x > \rho(t)$ , then  $u(x, t) > \varphi(x)$  and in this region  $u(x, t)$  is a classical solution to the equation

$$-\frac{\partial u}{\partial t} - \frac{\sigma(t)^2}{2} \frac{\partial^2 u}{\partial x^2} - \left( r(t) - q(t) - \frac{\sigma(t)^2}{2} \right) \frac{\partial u}{\partial x} + r(t)u = 0.$$

**Remark 10.** It is easy to show  $\rho(T) = \ln E \min[1, r(T)/q(T)]$ , using the way of [10].

**Theorem 9** Suppose that  $u(x, t)$  is the viscosity solution of (18). Assume that  $0 < \alpha \leq 1$ ,  $r(t)/\sigma^2(t)$  is increasing and  $q(t)/\sigma^2(t)$  decreasing on  $t$ . Then we have

- (i)  $u_{\Delta x}(x, t)$  converges to  $u(x, t)$  as  $\Delta x \rightarrow 0$ .
- (ii)  $\rho_{\Delta x}(t)$  converges to  $\rho(t)$  as  $\Delta x \rightarrow 0$ .

**Proof** Suppose that  $u(x, t)$  is the viscosity solution of (18) and denote

$$\begin{aligned} u^*(x, t) &= \lim_{\Delta x \rightarrow 0} \sup_{(y, s) \rightarrow (x, t)} u_{\Delta x}(y, s), \\ u_*(x, t) &= \lim_{\Delta x \rightarrow 0} \inf_{(y, s) \rightarrow (x, t)} u_{\Delta x}(y, s) \end{aligned} \quad (39)$$

From (36),  $u^*$  and  $u_*$  are well defined and we have  $0 \leq u_*(x, t) \leq u^*(x, t) \leq E$ . Obviously  $u^* \in USC([0, T] \times R)$  and  $u_* \in LSC([0, T] \times R)$ . If we prove that  $u^*$  and  $u_*$  are subsolution and supersolution of (18), respectively, then from lemma 7, we have  $u^* \leq u_*$  and thus  $u^* = u_* = u(x, t)$  becomes a viscosity solution of (18), and therefore we have the convergence of the approximated solution  $u_{\Delta x}(x, t)$ .

We will prove that  $u^*$  is a subsolution of (18). (The fact that  $u_*$  is a supersolution is similarly proved.) From the definition (35), we can easily know that

$$u^*(x, T) = \varphi(x) = (E - e^x)^+.$$

Suppose that  $\phi \in C^{1,2}([0, T] \times R)$  and  $u^* - \phi$  attains a local maximum at  $(x_0, t_0) \in [0, T] \times R$ . We might as well assume that  $(u^* - \phi)(x_0, t_0) = 0$  and  $(x_0, t_0)$  is a strict local maximum on  $B_r = \{(x, t) : t_0 \leq t \leq t_0 + r, |x - x_0| \leq r\}, r > 0$ . Let  $\Phi = \phi - \epsilon, \epsilon > 0$ , then  $u^* - \Phi$  attains a strict local maximum at  $(x_0, t_0)$  and

$$(u^* - \Phi)(x_0, t_0) > 0. \quad (40)$$

From the definition of  $u^*$ , there exists a sequence  $u_{\Delta x_k}(y_k, s_k)$  such that  $\Delta x_k \rightarrow 0, y_k \rightarrow x_0, s_k \rightarrow t_0$  and

$$\lim_{k \rightarrow \infty} u_{\Delta x_k}(y_k, s_k) = u^*(x_0, t_0). \quad (41)$$

If we denote the global maximum point of  $u_{\Delta x_k} - \Phi$  on  $B_r$  by  $(\hat{y}_k, \hat{S}_k)$ , then there exists a subsequence  $u_{\Delta x_{k_i}}(\hat{y}_{k_i}, \hat{S}_{k_i})$  such that

$$\Delta x_{k_i} \rightarrow 0, (\hat{y}_{k_i}, \hat{S}_{k_i}) \rightarrow (x_0, t_0), (u_{\Delta x_{k_i}} - \Phi)(\hat{y}_{k_i}, \hat{S}_{k_i}) \rightarrow (u^* - \Phi)(x_0, t_0) \text{ as } k_i \rightarrow \infty. \quad (42)$$

Indeed, suppose  $(\hat{y}_{k_i}, \hat{S}_{k_i}) \rightarrow (\hat{y}, \hat{s})$ , then from (41) we have

$$(u^* - \Phi)(x_0, t_0) = \lim_{k_i \rightarrow \infty} (u_{\Delta x_{k_i}} - \Phi)(y_{k_i}, s_{k_i}) \leq \lim_{k_i \rightarrow \infty} (u_{\Delta x_{k_i}} - \Phi)(\hat{y}_{k_i}, \hat{S}_{k_i}) \leq (u^* - \Phi)(\hat{y}, \hat{s}).$$

Therefore we have  $(\hat{y}, \hat{s}) = (x_0, t_0)$ , since  $(x_0, t_0)$  is a strict local maximum of  $(u^* - \Phi)$ . Thus for sufficiently large  $k_i$ ,  $\Delta t = \Delta t(\Delta x_{k_i})$  defined by (37) is small enough and if  $(x, \hat{S}_{k_i} + \Delta t(\Delta x_{k_i})) \in B_r$ , then we have

$$(u_{\Delta x_{k_i}} - \Phi)(x, \hat{S}_{k_i} + \Delta t(\Delta x_{k_i})) \leq (u_{\Delta x_{k_i}} - \Phi)(\hat{y}_{k_i}, \hat{S}_{k_i}),$$

that is ,

$$u_{\Delta x_{k_i}}(x, \hat{S}_{k_i} + \Delta t(\Delta x_{k_i})) \leq \Phi(x, \hat{S}_{k_i} + \Delta t(\Delta x_{k_i})) + (u_{\Delta x_{k_i}} - \Phi)(\hat{y}_{k_i}, \hat{S}_{k_i}). \quad (43)$$

From (40) and (42), we have

$$(u_{\Delta x_{k_i}} - \Phi)(\hat{y}_{k_i}, \hat{S}_{k_i}) > 0 \text{ (for sufficiently large } k_i). \quad (44)$$

For every  $k_i$ , define  $t_n$  and  $\Delta t_n = \frac{\alpha \Delta x_{k_i}^2}{\sigma^2(t_n)}$  as in (19) with  $\Delta x_{k_i}$ . Select  $t_{n_i}$  and  $j_{k_i} = j$  such that  $\hat{y}_{k_i} \in [(j-1/2)\Delta x_{k_i} + c, (j+1/2)\Delta x_{k_i} + c)$ ,  $\hat{S}_{k_i} \in [t_{n_i}, t_{n_i+1})$ , and simply denote  $t_{n_i} = t_n$  and  $j = j_{k_i}$ . Then from (38), lemma 4 with (43) and lemma 5 we have

$$\begin{aligned} u_{\Delta x_{k_i}}(\hat{y}_{k_i}, \hat{S}_{k_i}) &= U_j^n = (\mathbf{F}_n \mathbf{U}^{n+1})_j = [\mathbf{F}_n \mathbf{u}_{\Delta t_{k_i}}(\bullet, \hat{S}_{k_i} + \Delta t(\Delta x_{k_i}))](\hat{y}_{k_i}) \\ &\leq \{\mathbf{F}_n[\Phi(\bullet, \hat{S}_{k_i} + \Delta t) + (u_{\Delta x_{k_i}} - \Phi)(\hat{y}_{k_i}, \hat{S}_{k_i})]\}(\hat{y}_{k_i}) \\ &\leq \{\mathbf{F}_n[\Phi(\bullet, \hat{S}_{k_i} + \Delta t)]\}(\hat{y}_{k_i}) + (u_{\Delta x_{k_i}} - \Phi)(\hat{y}_{k_i}, \hat{S}_{k_i}). \end{aligned}$$

Thus we have

$$\Phi(\hat{y}_{k_i}, \hat{S}_{k_i}) - \{\mathbf{F}_n[\Phi(\bullet, \hat{S}_{k_i} + \Delta t)]\}(\hat{y}_{k_i}) \leq 0.$$

Therefore using (26) we have

$$\begin{aligned} &\Phi(\hat{y}_{k_i}, \hat{S}_{k_i}) - \max \left\{ \frac{1}{1 + r_n \Delta t_n} \left[ \left( 1 - \frac{\sigma_n^2 \Delta t_n}{\Delta x_{k_i}} \right) \Phi(\hat{y}_{k_i}, \hat{S}_{k_i} + \Delta t(\Delta x_{k_i})) + \right. \right. \\ &\quad \left. \left. + \frac{\sigma_n^2 \Delta t_n}{\Delta x_{k_i}} [a_n \Phi(\hat{y}_{k_i} + \Delta x_{k_i}, \hat{S}_{k_i} + \Delta t) + (1 - a_n) \Phi(\hat{y}_{k_i} - \Delta x_{k_i}, \hat{S}_{k_i} + \Delta t)] \right], \varphi_j \right\} \leq 0. \end{aligned}$$

This inequality is equivalent to the following.

$$\begin{aligned} &\min \left\{ \frac{\Delta t_n}{1 + r_n \Delta t_n} \left[ \frac{\Phi(\hat{y}_{k_i}, \hat{S}_{k_i}) - \Phi(\hat{y}_{k_i}, \hat{S}_{k_i} + \Delta t(\Delta x_{k_i}))}{\Delta t_n} - \right. \right. \\ &\quad - \frac{\sigma_n^2}{2} \frac{\Phi(\hat{y}_{k_i} + \Delta x_{k_i}, \hat{S}_{k_i} + \Delta t) - 2\Phi(\hat{y}_{k_i}, \hat{S}_{k_i} + \Delta t) + \Phi(\hat{y}_{k_i} - \Delta x_{k_i}, \hat{S}_{k_i} + \Delta t)}{2\Delta x_{k_i}} - \\ &\quad - \left( r_n - q_n - \frac{\sigma_n^2}{2} \right) \frac{\Phi(\hat{y}_{k_i} + \Delta x_{k_i}, \hat{S}_{k_i} + \Delta t) - \Phi(\hat{y}_{k_i} - \Delta x_{k_i}, \hat{S}_{k_i} + \Delta t)}{2\Delta x_{k_i}} + \\ &\quad \left. \left. + r_n \Phi(\hat{y}_{k_i}, \hat{S}_{k_i}) \right], \Phi(\hat{y}_{k_i}, \hat{S}_{k_i}) - \varphi_j \right\} \leq 0. \end{aligned}$$

Noting that  $\frac{\Delta t_n}{1 + r_n \Delta t_n} > 0$ , we have

$$\begin{aligned} &\min \left\{ \frac{\Phi(\hat{y}_{k_i}, \hat{S}_{k_i}) - \Phi(\hat{y}_{k_i}, \hat{S}_{k_i} + \Delta t(\Delta x_{k_i}))}{\Delta t_n} - \right. \\ &\quad - \frac{\sigma_n^2}{2} \frac{\Phi(\hat{y}_{k_i} + \Delta x_{k_i}, \hat{S}_{k_i} + \Delta t) - 2\Phi(\hat{y}_{k_i}, \hat{S}_{k_i} + \Delta t) + \Phi(\hat{y}_{k_i} - \Delta x_{k_i}, \hat{S}_{k_i} + \Delta t)}{2\Delta x_{k_i}} - \\ &\quad - \left( r_n - q_n - \frac{\sigma_n^2}{2} \right) \frac{\Phi(\hat{y}_{k_i} + \Delta x_{k_i}, \hat{S}_{k_i} + \Delta t) - \Phi(\hat{y}_{k_i} - \Delta x_{k_i}, \hat{S}_{k_i} + \Delta t)}{2\Delta x_{k_i}} + \\ &\quad \left. + r_n \Phi(\hat{y}_{k_i}, \hat{S}_{k_i}) - \varphi_j \right\} \leq 0. \end{aligned}$$

This inequality is equivalent to the following.

$$\min \left\{ \left[ \frac{\Phi(\hat{y}_{k_i}, \hat{S}_{k_i}) - \Phi(\hat{y}_{k_i}, \hat{S}_{k_i} + \Delta t(\Delta x_{k_i}))}{\Delta t(\Delta x_{k_i})} \right] \frac{\Delta t(\Delta x_{k_i})}{\Delta t_n} - \frac{\frac{\sigma_n^2}{2} \Phi(\hat{y}_{k_i} + \Delta x_{k_i}, \hat{S}_{k_i} + \Delta t) - 2\Phi(\hat{y}_{k_i}, \hat{S}_{k_i} + \Delta t) + \Phi(\hat{y}_{k_i} - \Delta x_{k_i}, \hat{S}_{k_i} + \Delta t)}{2\Delta x_{k_i}} - \left( r_n - q_n - \frac{\sigma_n^2}{2} \right) \frac{\Phi(\hat{y}_{k_i} + \Delta x_{k_i}, \hat{S}_{k_i} + \Delta t) - \Phi(\hat{y}_{k_i} - \Delta x_{k_i}, \hat{S}_{k_i} + \Delta t)}{2\Delta x_{k_i}} + r_n \Phi(\hat{y}_{k_i}, \hat{S}_{k_i}) - \varphi_j \right\} \leq 0.$$

Let  $k_i \rightarrow \infty$ , then  $\Delta x_{k_i} \rightarrow 0$ . From lemma 6, we have  $\frac{\Delta t(\Delta x_{k_i})}{\Delta t_n} \rightarrow 1$ . Thus we have

$$\min \left\{ -\frac{\partial \Phi}{\partial t} - \frac{\sigma^2(t)}{2} \frac{\partial^2 \Phi}{\partial x^2} - \left( r(t) - q(t) - \frac{\sigma^2(t)}{2} \right) \frac{\partial \Phi}{\partial x} + r(t)\Phi, \Phi - \varphi \right\}_{(x_0, t_0)} \leq 0.$$

(Here we considered  $(\hat{y}_{k_i}, \hat{S}_{k_i}) \rightarrow (x_0, t_0)$  and  $\varphi_{j_{k_i}} \rightarrow \varphi(x_0)$ . Let  $\epsilon \rightarrow 0$ , then we have

$$\min \left\{ -\frac{\partial \phi}{\partial t} - \frac{\sigma^2(t)}{2} \frac{\partial^2 \phi}{\partial x^2} - \left( r(t) - q(t) - \frac{\sigma^2(t)}{2} \right) \frac{\partial \phi}{\partial x} + r(t)\phi, \phi - \varphi \right\}_{(x_0, t_0)} \leq 0.$$

Since  $u^*(x_0, t_0) = \phi(x_0, t_0)$ ,  $u^*$  is a subsolution of (18). Thus we proved (i).

Now we will prove  $\rho_{\Delta x}(t)$  converges to  $\rho(t)$  as  $\Delta x \rightarrow 0$ . The main idea is from [12]. First, from Corollary of theorem 7 and Remark 9, we have

$$\lim_{\Delta x \rightarrow 0} \rho_{\Delta x}(t_{N-1}) = \ln \min \left[ E, \frac{r(T)}{q(T)} E \right] = \rho(T).$$

Now fix  $t_0 \in [0, T)$  and suppose  $x < \lim_{\Delta x \rightarrow 0} \sup \rho_{\Delta x}(t_0)$ . Then there exists a sequence  $\Delta x_k$  such that  $\Delta x_k \rightarrow 0$  and  $\lim_{k \rightarrow \infty} \rho_{\Delta x_k}(t_0) > x$ . Denote by  $\{t_n^{(k)}\}$  the time partition corresponding to  $\Delta x_k$  and let  $t_0 \in [t_{n_0-1}^{(k)}, t_{n_0}^{(k)})$ . Then  $t_{n_0}^{(k)} \rightarrow t_0$  as  $k \rightarrow \infty$ . Since  $\rho_{\Delta x}$  is increasing, we have

$$\rho_{\Delta x_k}(t_{n_0}^{(k)}) \geq \rho_{\Delta x_k}(t_0).$$

Select  $x_k$  such that  $\lim \Delta x_k = x$ . For sufficiently large  $k$ ,  $x_k < \rho_{\Delta x_k}(t_{n_0}^{(k)}) = j_{n_0} \Delta x_k + c$  and thus we have  $u_{\Delta x_k}(x_k, t_{n_0}^{(k)}) = \varphi(x_k)$  (since  $x_k$  is in the exercise region). Thus we have  $\Delta x_k \rightarrow 0 \Rightarrow u(x, t_0) = \varphi(x) \Rightarrow x \leq \rho(t_0)$ , so we have

$$\limsup \rho_{\Delta x}(t_0) \leq \rho(t_0).$$

Now we will prove  $\liminf \rho_{\Delta x}(t_0) \geq \rho(t_0)$ . Assume that there exists  $\epsilon > 0$  such that  $\liminf \rho_{\Delta x}(t_0) < \rho(t_0) - 2\epsilon$ . From the fact that  $\rho(t)$  is continuous, there exists  $\delta > 0$  such that  $\liminf \rho_{\Delta x}(t_0) < \rho(t) - 2\epsilon, t \in [t_0 - \delta, t_0 + \delta]$ . Therefore there exists a sequence  $\Delta x_k \rightarrow 0$  such that  $\rho_{\Delta x_k}(t_0) < \rho(t) - 2\epsilon, t \in [t_0 - \delta, t_0 + \delta]$ . Now let  $\underline{\rho} := \min\{\rho(t) : t \in [t_0 - \delta, t_0 + \delta]\}$  and  $Q = \{(x, t) : t \in (t_0 - \delta, t_0], x \in (\underline{\rho} - 2\epsilon, \underline{\rho} - \epsilon)\}$ . Then since  $\rho_{\Delta x_k}(t)$  is increasing, we have  $\rho_{\Delta x_k}(t) \leq \rho_{\Delta x_k}(t_0) \leq \underline{\rho} - 2\epsilon < x < \underline{\rho} - \epsilon$  for  $(x, t) \in Q$ . Therefore we have

$$\rho_{\Delta x_k}(t) < x < \rho(t) - \epsilon, (x, t) \in Q. \quad (45)$$

From  $\rho_{\Delta x_k}(t) < x$ , let  $u_{\Delta x_k}(x, t) = U_j^n$ , then we have

$$-\frac{U_j^{n+1} - U_j^n}{\Delta t} - \frac{\sigma^2(t_n)}{2} \frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{\Delta x^2} - \left[ r(t_n) - q(t_n) - \frac{\sigma^2(t_n)}{2} \right] \frac{U_{j+1}^{n+1} - U_{j-1}^{n+1}}{2\Delta x} + r(t_n)U_j^n = 0.$$

Letting  $\Delta x_k \rightarrow 0$ , then  $u_{\Delta x_k}(x, t) (= U_j^n)$  converges to the viscosity solution  $u(x, t)$  and from the regularity of viscosity solution (Theorem 8), we have

$$\frac{\partial u}{\partial t} + \frac{\sigma(t)^2}{2} \frac{\partial^2 u}{\partial x^2} + \left( r(t) - q(t) - \frac{\sigma(t)^2}{2} \right) \frac{\partial u}{\partial x} - r(t)u = 0, \quad (x, t) \in Q. \quad (46)$$

On the other hand, from (45), we have  $x < \rho(t) - \epsilon$ , thus  $x$  is in the exercise region of (18) and  $u(x, t) = E - e^x$ . So we have

$$\frac{\partial u}{\partial t} + \frac{\sigma(t)^2}{2} \frac{\partial^2 u}{\partial x^2} + \left( r(t) - q(t) - \frac{\sigma(t)^2}{2} \right) \frac{\partial u}{\partial x} - r(t)u = q(t)e^x - r(t)E, \quad (x, t) \in Q.$$

This contradicts to (46). Thus we proved

$$\liminf \rho_{\Delta x}(t_0) \geq \rho(t_0) \geq \limsup \rho_{\Delta x}(t_0).$$

So we have  $\lim \rho_{\Delta x}(t_0) = \rho(t_0)$ . (QED)

**Corollary 1** (Monotonicity of American put option's price and optimal exercise boundary) *Suppose that  $u(x, t)$  is the viscosity solution of (18). Assume that  $r(t)/\sigma^2(t)$  is increasing and  $q(t)/\sigma^2(t)$  is decreasing on  $t$ . Then we have*

- (i)  $u(x, t)$  is decreasing on  $x$  and  $t$ .
- (ii)  $\rho(t)$  is increasing on  $t$ .

(Proof) (i) comes from Theorem 4 (ii), Theorem 5 and Theorem 9 (i).

(ii) comes from (ii) of Corollary of Theorem 7 and Theorem 9 (ii). (QED)

**Lemma 8** [12] *Let  $\Omega \subset R^m$  and  $f_n(x_1, \dots, x_m)$  be pointwise convergent to a continuous function  $f(x_1, \dots, x_m)$ . Assume that  $f_n$  and  $f$  are monotone on  $\Omega$ . Then  $f_n$  uniformly converges to  $f$  on any compact subset of  $\Omega$ .*

**Theorem 10** *When  $\Delta x \rightarrow 0$ , then  $u_{\Delta x}(x, t)$  uniformly converges to  $u(x, t)$  on any bounded subdomain of  $[0, T] \times R$  and  $\rho_{\Delta x}(t)$  uniformly converges to  $\rho(t)$ .*

(Proof) From the result of Theorem 10,  $u(x, t)$  and  $\rho(t)$  are both monotone. Thus from lemma 8, we have the desired results. (QED)

**Remark 11.** Consider the convergence of the binomial tree methods. As shown in section 3, in the lattice  $Q_c = \{(x_j, t_n) : x_j = j\Delta x + c, 0 \leq n \leq N-1, j \in Z\}$ , the

explicit difference scheme (27) with  $\alpha = \sigma_n^2 \Delta t_n / \Delta x^2 = 1$  coincides with BTM in the sense of neglecting  $O(\Delta x^3)$ .

Let  $U_j^n$  be the prices by the explicit difference scheme (27) and  $V_j^n$  the BTM prices. Then we have

$$\begin{aligned} U_j^n &= \max \left\{ \frac{1}{\rho_n} [a_n U_{j+1}^{n+1} + (1 - a_n) U_{j-1}^{n+1}], (E - S_0 e^{j\Delta x})^+ \right\}, \\ V_j^n &= \max \left\{ \frac{1}{\rho_n} [\theta_n V_{j+1}^{n+1} + (1 - \theta_n) V_{j-1}^{n+1}], (E - S_0 e^{j\Delta x})^+ \right\}. \end{aligned}$$

Let  $u = e^{\Delta x}$ . Then we have

$$\begin{aligned} |V_j^n - U_j^n| &\leq \frac{1}{\rho_n} [|\theta_n V_{j+1}^{n+1} - a_n U_{j+1}^{n+1}| + |(1 - \theta_n) V_{j-1}^{n+1} - (1 - a_n) U_{j-1}^{n+1}|] \\ &\leq \frac{1}{\rho_n} [\theta_n |V_{j+1}^{n+1} - U_{j+1}^{n+1}| + |\theta_n - a_n| U_{j+1}^{n+1} + (1 - \theta_n) |V_{j-1}^{n+1} - U_{j-1}^{n+1}| + |\theta_n - a_n| U_{j-1}^{n+1}] \\ &\leq \frac{1}{\rho_n} [\|\mathbf{V}^{n+1} - \mathbf{U}^{n+1}\|_{l^\infty(Z)} + 2O(\Delta x^3) \|\mathbf{U}^{n+1}\|_{l^\infty(Z)}]. \end{aligned}$$

Here we considered lemma 3:  $\theta_n = a_n + O(\Delta x^3)$ . Note that  $\|\mathbf{U}^{n+1}\|_{l^\infty(Z)} \leq E$ , then we have

$$|V_j^n - U_j^n| \leq \frac{1}{\rho_n} [\|\mathbf{V}^{n+1} - \mathbf{U}^{n+1}\|_{l^\infty(Z)} + 2E \cdot O(\Delta x^3)].$$

Since  $r_n / \sigma_n^2$  is increasing, we have

$$\exp(-r_n \Delta t_n) = \exp(-r_n \Delta x^2 / \sigma_n^2) \leq \exp(-r_0 \Delta x^2 / \sigma_0^2) =: A,$$

and using  $\rho_n^{-1} = (1 + r_n \Delta t_n) = \exp(-r_n \Delta t_n) + O(\Delta t_n^2)$ , then we have

$$|V_j^n - U_j^n| \leq A \|\mathbf{V}^{n+1} - \mathbf{U}^{n+1}\|_{l^\infty(Z)} + 2AE \cdot O(\Delta x^3).$$

Therefore we have

$$\begin{aligned} \|\mathbf{V}^n - \mathbf{U}^n\|_{l^\infty(Z)} &\leq A \|\mathbf{V}^{n+1} - \mathbf{U}^{n+1}\|_{l^\infty(Z)} + A2E \cdot O(\Delta x^3) \\ &\leq A^2 \|\mathbf{V}^{n+2} - \mathbf{U}^{n+2}\|_{l^\infty(Z)} + (A + A^2)2E \cdot O(\Delta x^3) \\ &\dots \dots \dots \\ &\leq A^{N-n} \|\mathbf{V}^N - \mathbf{U}^N\|_{l^\infty(Z)} + (A + A^2 + \dots + A^{N-n})2E \cdot O(\Delta x^3). \end{aligned}$$

Here noting that  $\|\mathbf{V}^N - \mathbf{U}^N\|_{l^\infty(Z)} = \max\{|V_j^N - U_j^N|\} = 0$ , we have

$$\|\mathbf{V}^n - \mathbf{U}^n\|_{l^\infty(Z)} \leq \frac{A}{1-A} 2E \cdot O(\Delta x^3) = O(\Delta x).$$

Thus from theorem 9, when  $\Delta x \rightarrow 0$ , the BTM prices  $V_j^n$  converges to the viscosity solution to the variational inequality (18).

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